Graph Modeling-Part1

CE642: Social and Economic Networks
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01

Random Walk

What is a Random Walk

- Given a graph and a starting point (node), we select a neighbor of it at random, and move to this neighbor;
- Then we select a neighbor of this node and move to it, and so on;
- The (random) sequence of nodes selected this way is a random walk on the graph

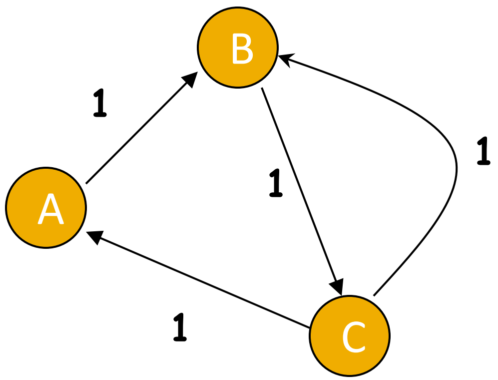
An example

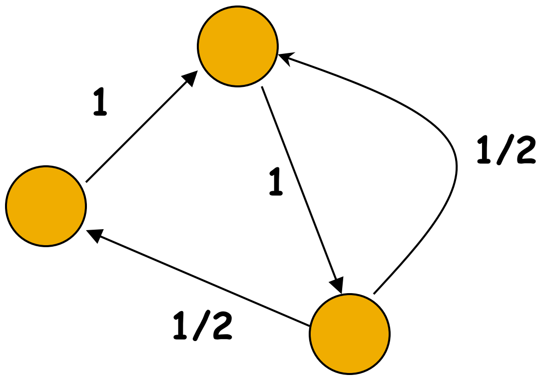
0	1	0
0	0	1
1	1	0

Adjacency matrix A

0	1	0
0	0	1
1/2	1/2	0

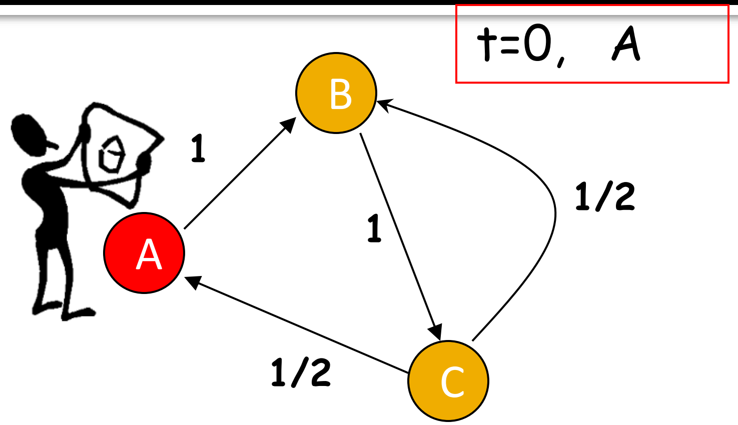
Transition matrix P





Slide from Purnamitra Sarkar, *Random Walks on Graphs: An Overview*

An example



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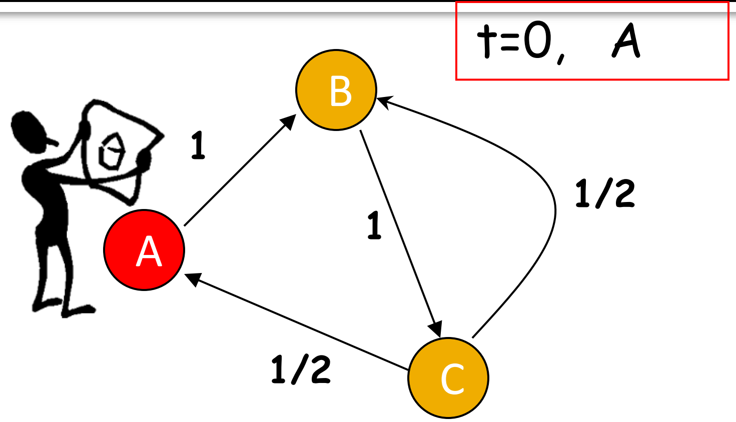

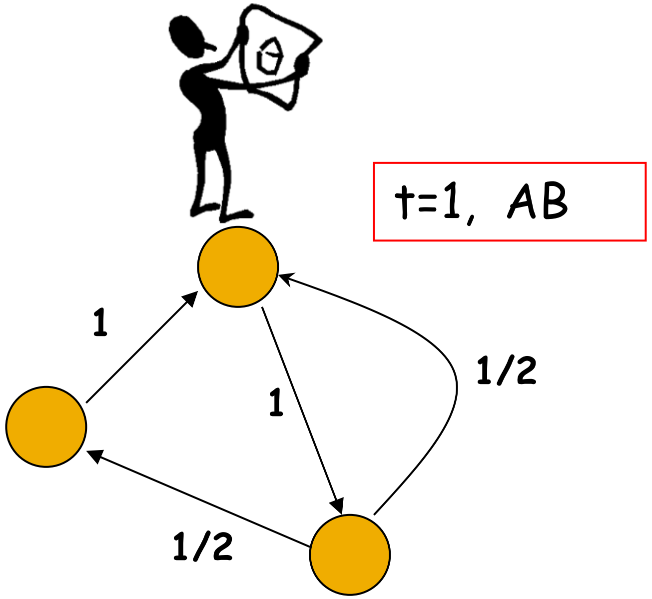

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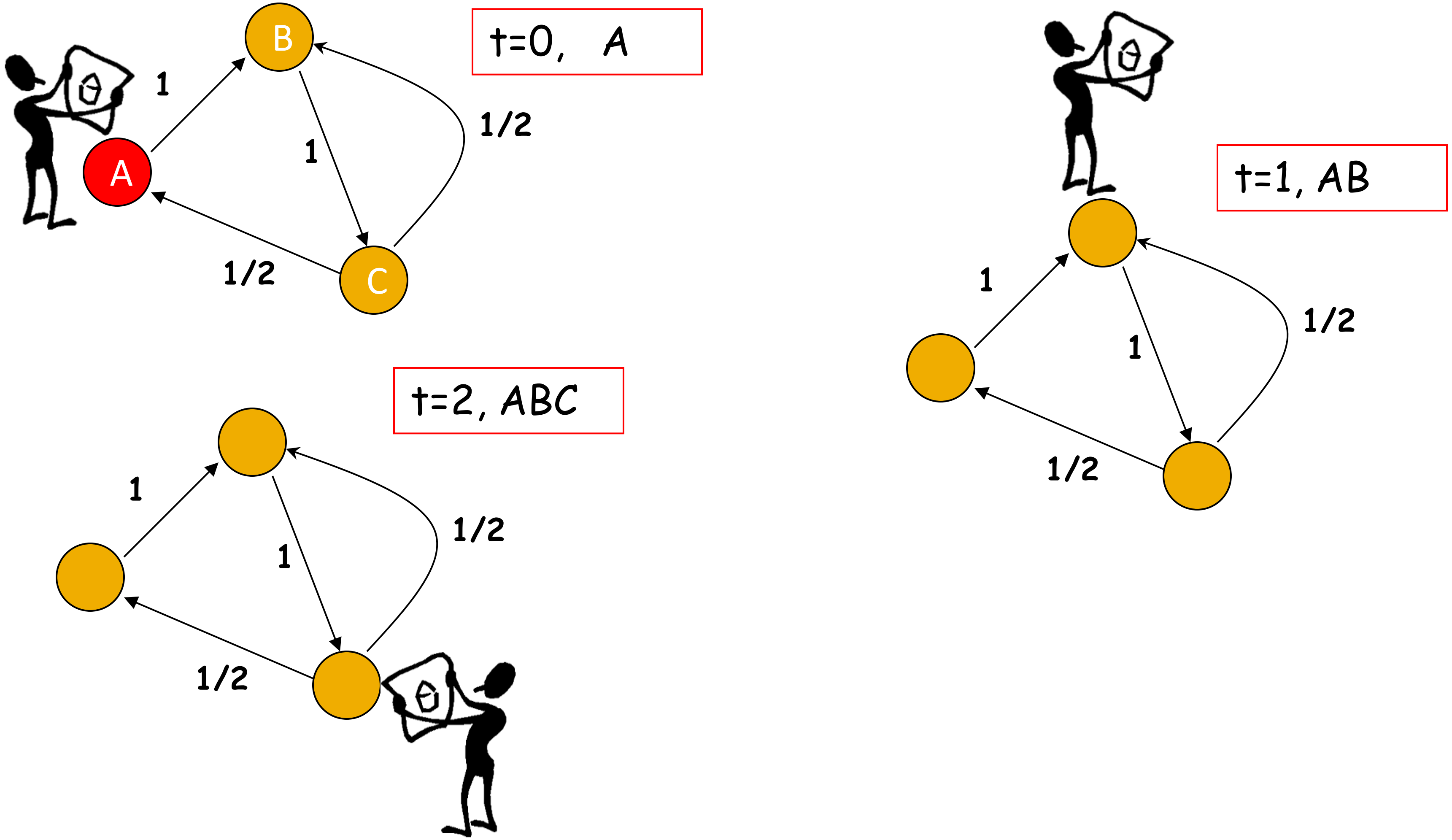
An example



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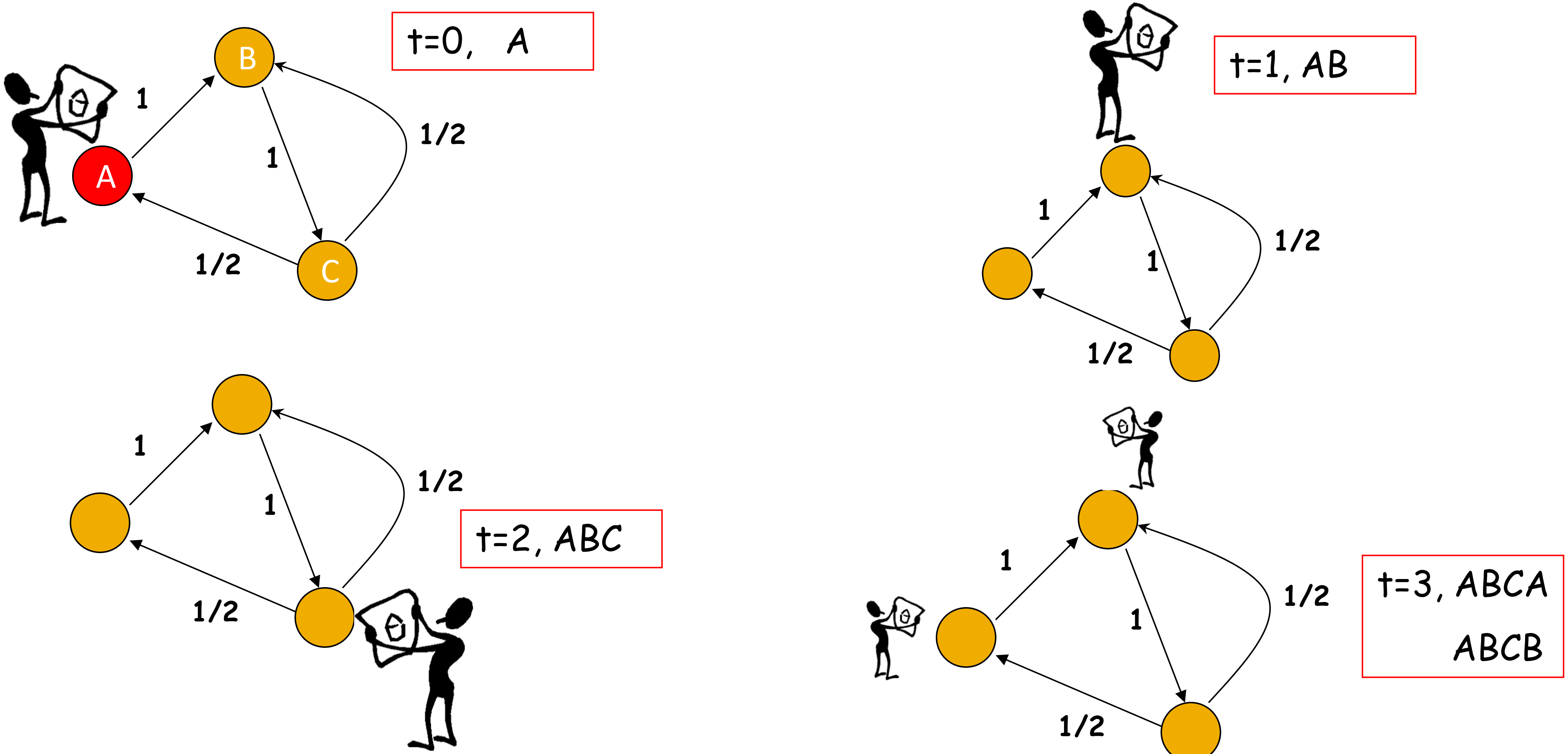
An example



Slide from Purnamitra Sarkar, *Random Walks on Graphs: An Overview*

Why are random walks interesting?

- When the underlying data has a natural graph structure, several physical processes can be conceived as a random walk

Data	Process
WWW	Random surfer
Internet	Routing
P2P	Search
Social network	Information percolation

Random walks: definitions

- nxn Adjacency matrix A.
  - A(i,j) = weight on edge from i to j
  - If the graph is undirected A(i,j)=A(j,i), i.e. A is symmetric
- nxn Transition matrix P.
  - P is row stochastic
  - P(i,j) = probability of stepping on node j from node i= A(i,j)/Σ**i**A(i,j)
- nxn Laplacian Matrix L.
  - L(i,j)=Σ**i**A(i,j)-A(i,j)=> L=D-A
  - **Symmetric positive semi-definite for undirected graphs??**
  - **Singular??**

Laplacian Matrix

- Positive semi-definite for undirected graphs.

$\forall \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}} \quad x^T L x \geq 0$

$$x^T L x = x^T D x - x^T A x$$

- $x^T D x = \sum_i \deg(i) x_i^2$
- $x^T A x = \sum_{i,j} A(i,j) x_i x_j$

$$x^T L x = \sum_i \deg(i) x_i^2 - \sum_{i,j} A(i,j) x_i x_j$$

$$x^T L x = \frac{1}{2} \sum_{i,j} A(i,j)(x_i - x_j)^2$$

$A(i,j) \geq 0 \Rightarrow x^T L x \geq 0$

Laplacian Matrix

- Singular

$$\mathbf{1} = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$$

$$L \cdot \mathbf{1} = (D - A)\mathbf{1} = D\mathbf{1} - A\mathbf{1}$$

- $D\mathbf{1} = \deg(i)$
- $A\mathbf{1} = \text{sum of neighbors} = \deg(i)$

$$L \cdot \mathbf{1} = 0$$

So, zero is the eigenvalue, and eigenvalues multiplication is the determinant. Therefore, det(L)=0.

Probability Distributions

- $x_t(i)$ = probability that the surfer is at node *i* at time *t*
- $x_{t+1}(i) = \sum_j$(Probability of being at node j)*Pr(j->i) $=\sum_j x_t(j) * P(j,i)$
- $x_{t+1} = x_t P = x_{t-1} * P * P = x_{t-2} * P * P * P = \ldots = x_0 P^t$

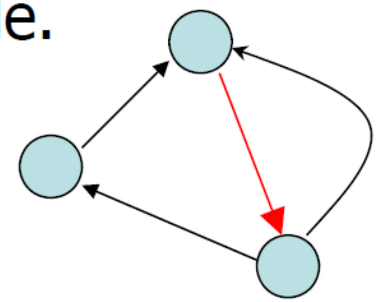
- What happens when the surfer keeps walking for a long time?
- Stationary distribution:
  - When the surfer keeps walking for a long time
  - When the distribution does not change anymore, i.e. $x_{T+1} = x_T$
  - For "well-behaved" graphs this does not depend on the start distribution!!!

What is a stationary distribution?

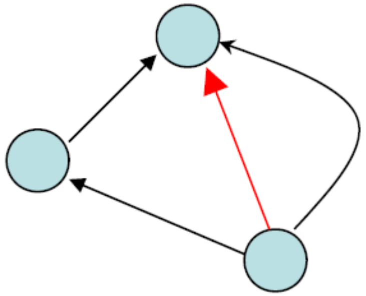
- The stationary distribution at a node is related to the amount of time a random walker spends visiting that node.
- Remember that we can write the probability distribution at a node as
  - $x_{t+1} = x_t P$
- For the stationary distribution v_0 we have
  - $v_0 = v_0 P$
- So, that's just the left eigenvector of the transition matrix!
- Interesting questions:
  - Does a stationary distribution always exist? Is it unique? (Yes, if the graph is "well-behaved")
  - What is "well-behaved"?
  - How fast will the random surfer approach this stationary distribution? (Mixing Time!)

Well-behaved graphs

- Irreducible: There is a path from every node to every other node.

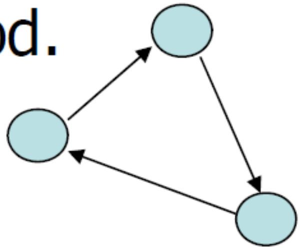


Irreducible

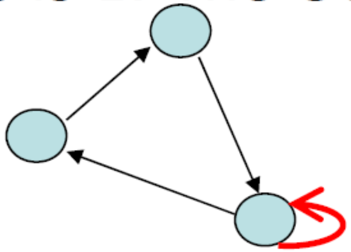


Not irreducible

- Aperiodic: The GCD of all cycle lengths is *1*. The GCD is also called period.



Periodicity is 3

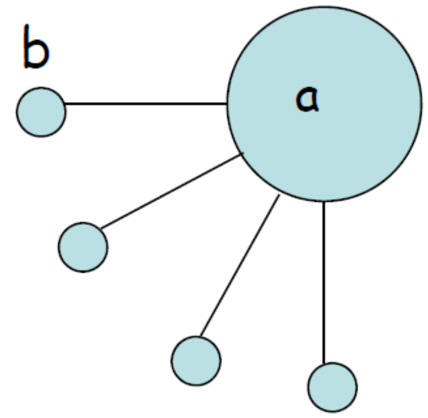


Aperiodic

Perron Frobenius Theorem

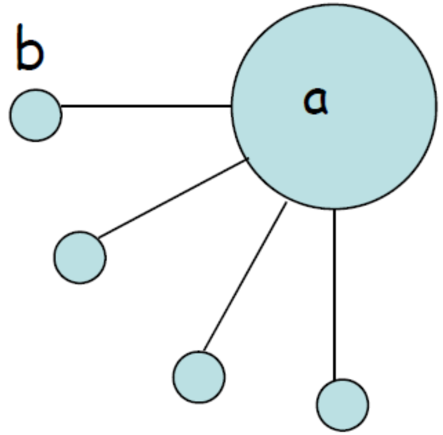
- If a markov chain is irreducible and aperiodic, then the largest eigenvalue of the transition matrix will be equal to **1** and all the other eigenvalues will be strictly less than **1**.
  - Let the eigenvalues of P be $\{\sigma_i | i=0:n-1\}$ in non-increasing order of σ_i .
  - $\sigma_0 = 1$ **>** $\sigma_1 > \sigma_2 >= ......>= \sigma_n$
- These results imply that for a well behaved graph there exists an unique stationary distribution.
- The pagerank uses these results.
- We know that
  - A connected undirected graph is irreducible
  - A connected non-bipartite undirected graph has a stationary distribution proportional to the degree distribution!
  - Makes sense, since larger the degree of the node more, likely a random walk is to come back to it.

Proximity measures from random walks



- How long does it take to hit node b in a random walk starting at node a ? Hitting time.
- How long does it take to hit node b and come back to node a ? Commute time.

Hitting and Commute times



- Hitting time from node i to node j
  - Expected number of hops to hit node j starting at node i
  - Is not symmetric. $h(a,b) \neq h(b,a)$
  - $h(i,j) = 1 + \Sigma_{k \in nbs(A)} p(i,k)h(k,j)$
- Commute time between node i and j
  - Is expected time to hit node j and come back to i
  - $c(i,j) = h(i,j) + h(j,i)$
  - Is symmetric. $c(a,b) = c(b,a)$

Random graphs

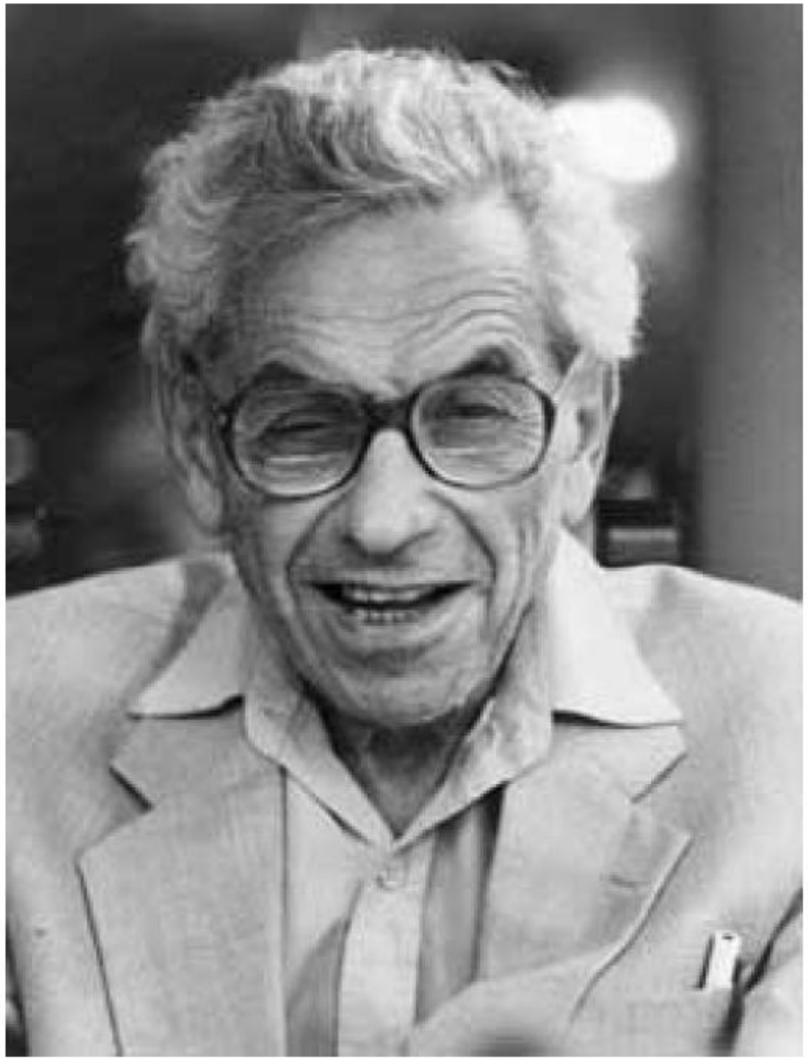
- A deterministic model D defines a single graph for each value of n (or t)

- A randomized model R defines a probability space $\langle G_n, P \rangle$ where G_n is the set of all graphs of size n, and P a probability distribution over the set G_n (similarly for t)
  - we call this a family of random graphs R, or a random graph R

02

Erdös-Renyi Random graphs

Erdös-Renyi Random graphs



Paul Erdös (1913-1996)

You may have heard about Erdös number!
What is your Erdös number?

Erdös-Renyi Random graphs

For generation of Erdös-Renyi network, one of the following methods is used:

1. The $G_{n,p}$ model
   - input: the number of vertices n, and a parameter p, $0 \le p \le 1$
   - process: for each pair (i,j), generate the edge (i,j) independently with probability p
2. Related, but not identical: The $G_{n,m}$ model
   - process: select m edges uniformly at random

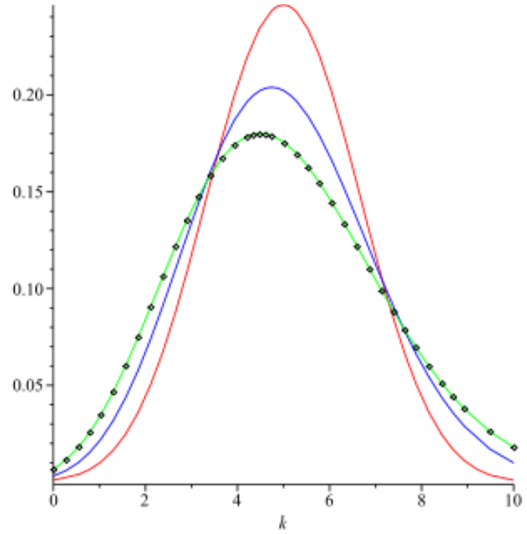
Erdös-Renyi Random graphs

- $G(n,p)$:
  - Consider a set of nodes $N = \{1,2,\dots,n\}$
  - Connect each pair i,j of nodes with probability p
  - The expected number of edges: $\binom{n}{2}p$
  - The expected degree of nodes: $(n-1)p$
- $G(n,M)$:
  - Choose M edges out of all $\binom{n}{2}$ pair of nodes: $\binom{\binom{n}{2}}{M}$ choices
  - Number of edges: M
  - The expected degree of nodes: $\frac{M}{\binom{n}{2}} \times (n-1) = \frac{2M}{n}$

Binomial Distribution

- Binomial Distribution:
  - Consider a sequence of Bernoulli trials. What is the probability of m heads out of n flips? $P(d)$ is:

$$\binom{n}{m} p^m (1-p)^{n-m}$$



  - Expected number of heads: *np*
  - The variance: *npq = np(1-p)*
  - Standard deviation: $\sqrt{np(1-p)}$
- Binomial distribution can be approximated by $\lambda = np$ for large n

$$P(d) \approx \frac{e^{-\lambda}\lambda^d}{d!}$$

- Highly concentrated around the mean, with a tail that drops exponentially

Poisson Networks

- Degree distribution: Binomial distribution
  - The probability of having d neighboring edges is equal to:

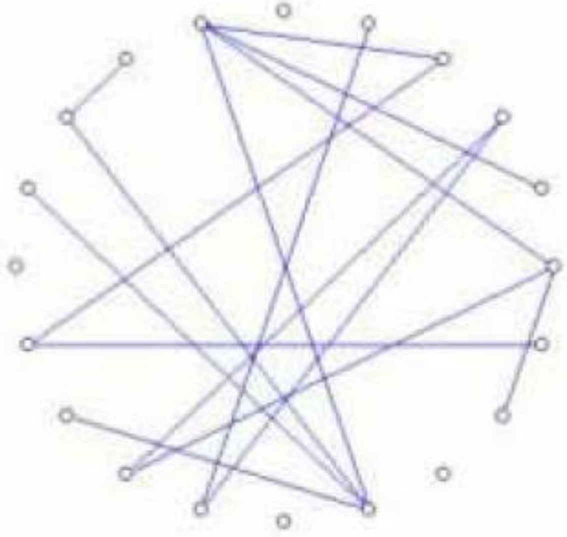
$$P(d) = \binom{n-1}{d} p^d (1-p)^{n-1-d}$$

- Can be approximated by $\lambda = (n-1)p = np$ for large n

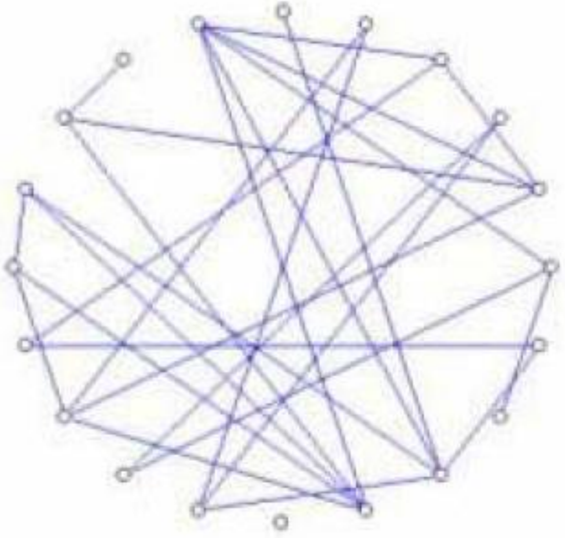
$$P(d) \approx \frac{e^{-\lambda}\lambda^d}{d!}$$

Example

$p = 0$
(a)



$p = 0.1$
(b)



$p = 0.2$
(c)

Clustering Coefficient

Let’s say a node v has degree k.

$$C_v = \frac{\text{number of links between neighbors of } v}{\binom{k}{2}}$$

$$\mathbb{E}[\text{edges among neighbors}] = \binom{k}{2} \cdot p$$

$$C_v = \frac{\binom{k}{2} \cdot p}{\binom{k}{2}} = p$$

Since $\langle k \rangle = (n-1)p \approx np$, we get:

$$p = \frac{\langle k \rangle}{n}$$

$$C = p \sim \frac{< k >}{n}$$

As n→∞, clustering goes to 0 in Poisson/Erdős–Rényi graphs, while **real-world networks** often **maintain high clustering**. That’s why we say:
"Erdős-Rényi graphs are poor models for social networks."

Diameter

- maximum length of shortest paths
  - To estimate the maximum distance between two nodes, we think:
    - Start from any node.
    - How many steps do we need until we can reach everyone?

$$\lambda^d = n \Rightarrow d = \log_\lambda n = \frac{\log n}{\log \lambda}$$

Phase transition

- Starting from some vertex v perform a BFS walk
- At each step of the BFS a Poisson process with mean λ, gives birth to new nodes
- When $\lambda < 1$ this process will stop after $O(\log n)$ steps
- When $\lambda > 1$, this process will continue for $O(n)$ steps

Are real-world networks random?

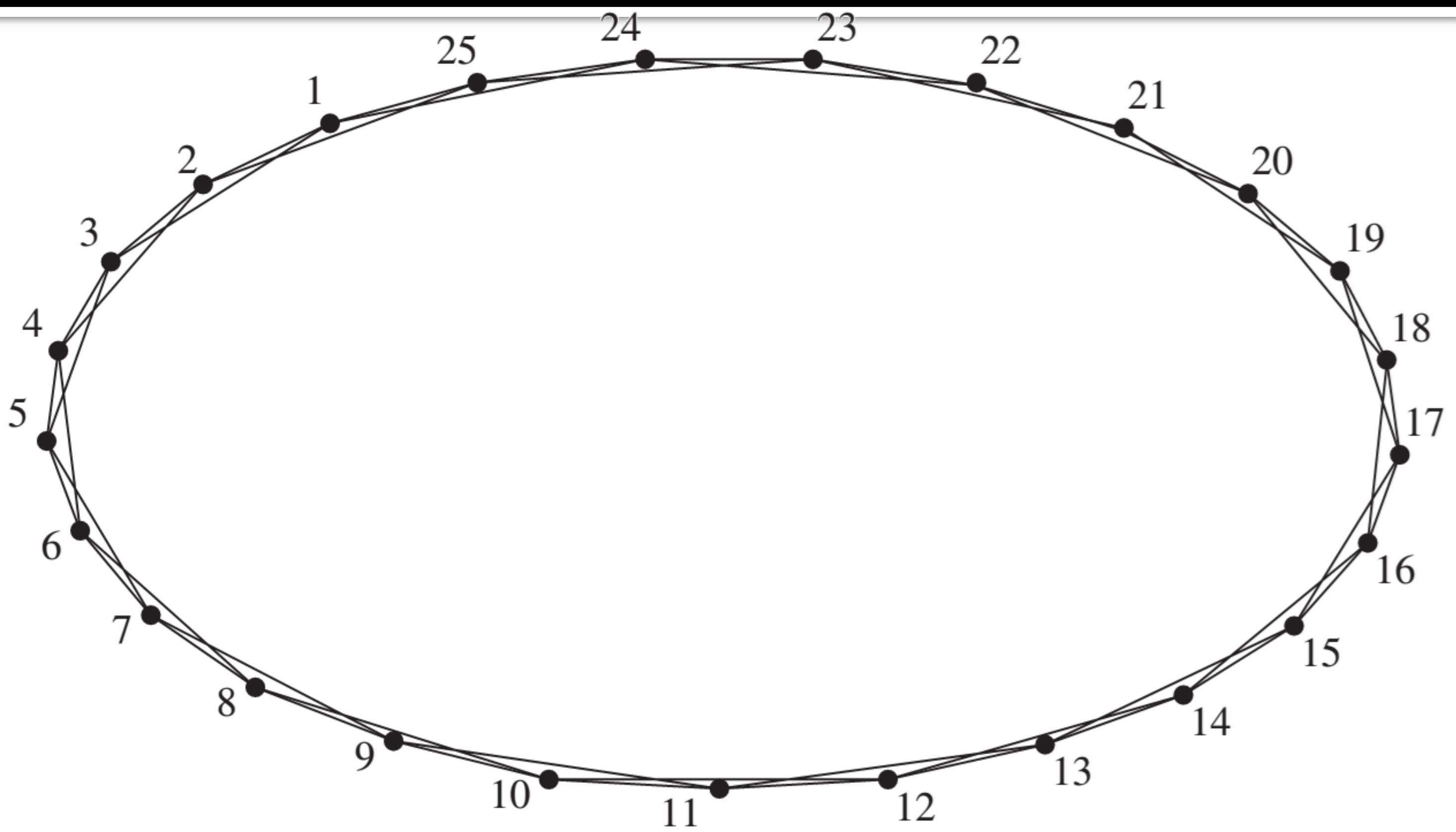
- A decade ago, the most elegant theory for modelling real-world networks was based on random graphs
- But real-world networks are not random (we will see)
- However, studies on random networks provides insights into complex structures

03

Watts-Strogatz Model

Watts-Strogatz Model

- Consider a n nodes cycle and connect each node to its $2m$ nearest nodes
- For m=2:
  - Diameter: $\frac{n}{4}$
  - Clustering Coefficient: $\frac{1}{2}$
- Diameter is high, while the clustering coefficient is also high

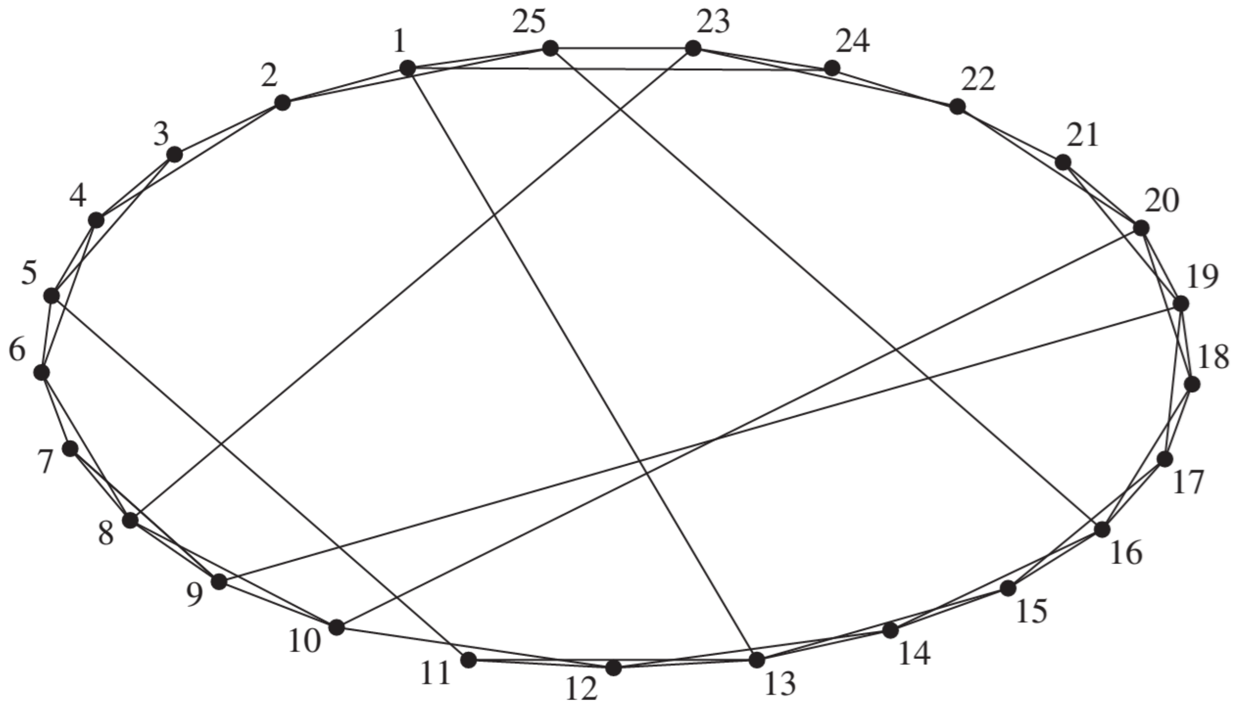


Global Clustering Coefficient

$$C = \frac{3 \times \text{number of triangles in the graph}}{\text{number of connected triples}}$$

Watts-Strogatz Model

- Watts & Strogatz show that with a few random rewiring the diameter will be decreased a lot.
- We will speak about **small-world** models deeply later

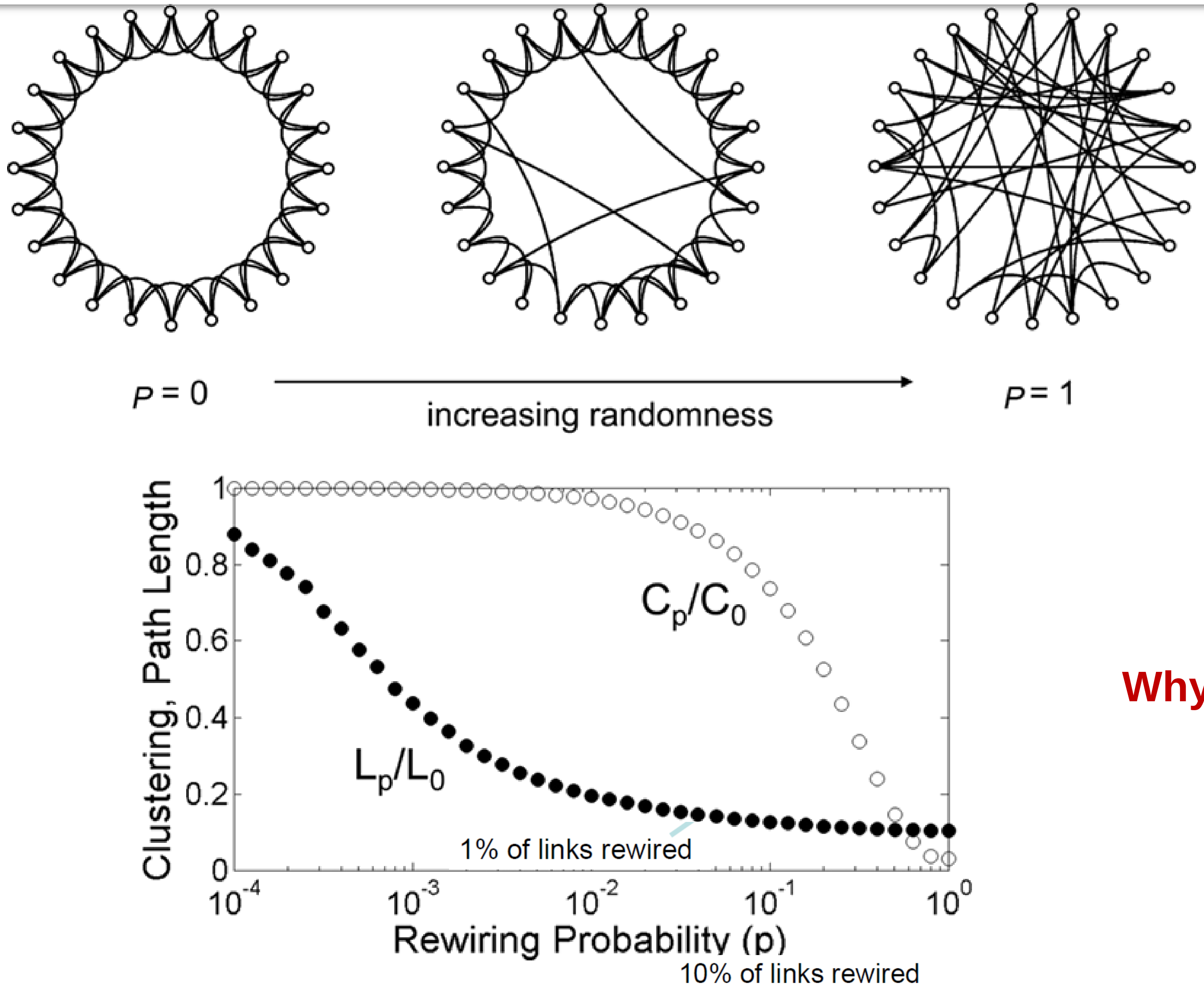


Watts-Strogatz Model

The construction algorithm:

- Consider a ring graph where each node is connected to its m nearest neighbors with undirected edges
- Choose a node and one of the edges that connects it to its nearest neighbors and then with probability P reconnect this edge to a node randomly chosen over the graph
  - provided that the duplication of edges and self-loops are forbidden
- The process is repeated until all nodes and nearest neighbor connecting edges are met
- Next, the edges that connect the nodes to their second-nearest neighbors are reconnected and the rewiring process is performed on them with the same conditions as above
- The same procedure is then repeated for the remaining edges connecting the nodes to their m nearest neighbors

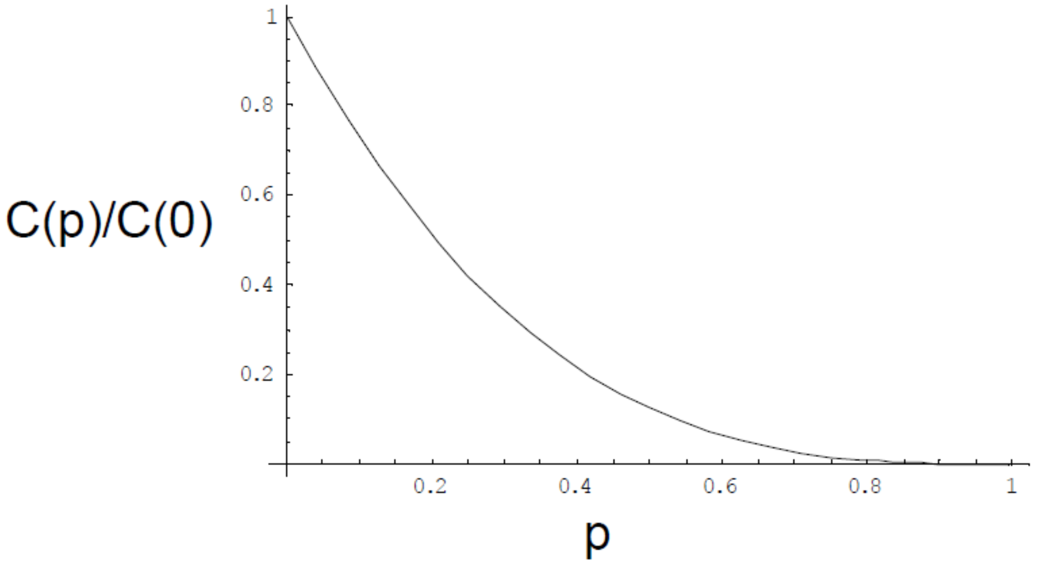
Watts-Strogatz Model

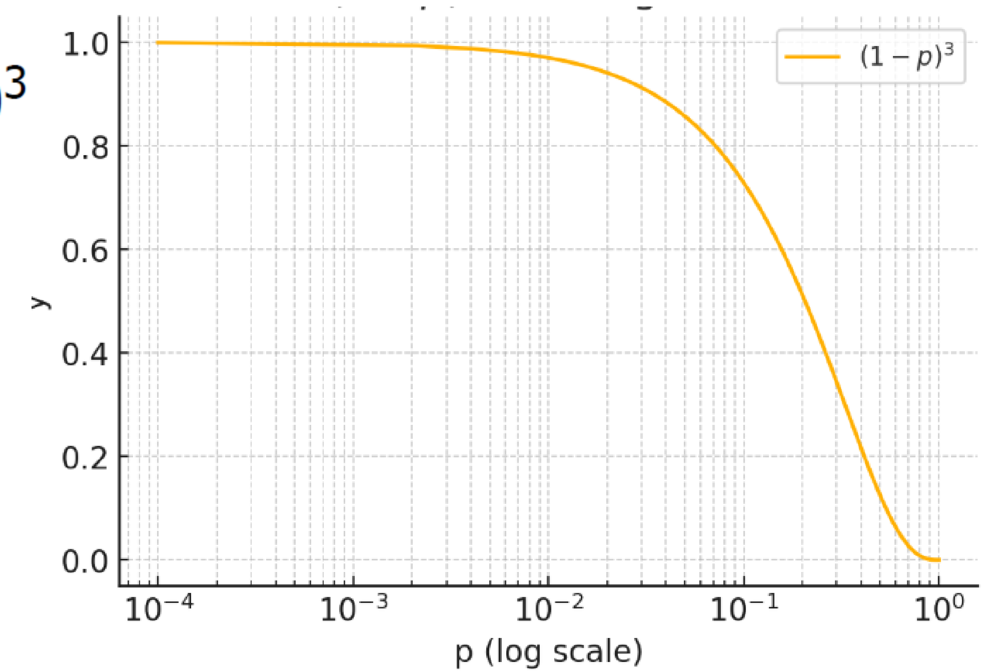


Why?

Clustering Coefficient

- The probability that a connected triple stays connected after rewiring
  - probability that none of the 3 edges were rewired $(1\text{-}p)^3$
  - probability that edges were rewired back to each other very small, can ignore
- Clustering coefficient = $C(p) = C(p{=}0)*(1\text{-}p)^3$

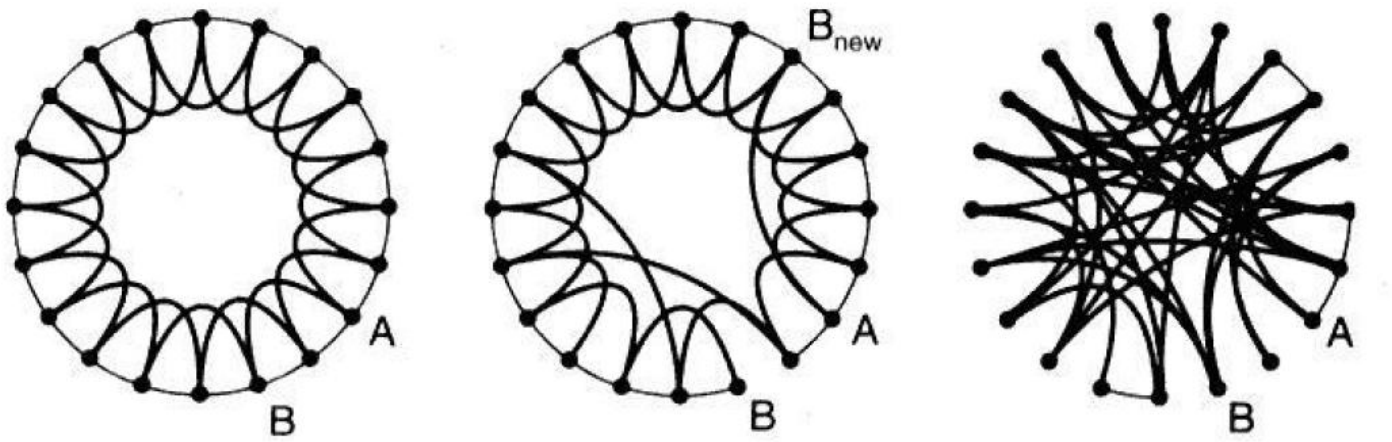




Watts-Strogatz Model

Reconciling two observations:

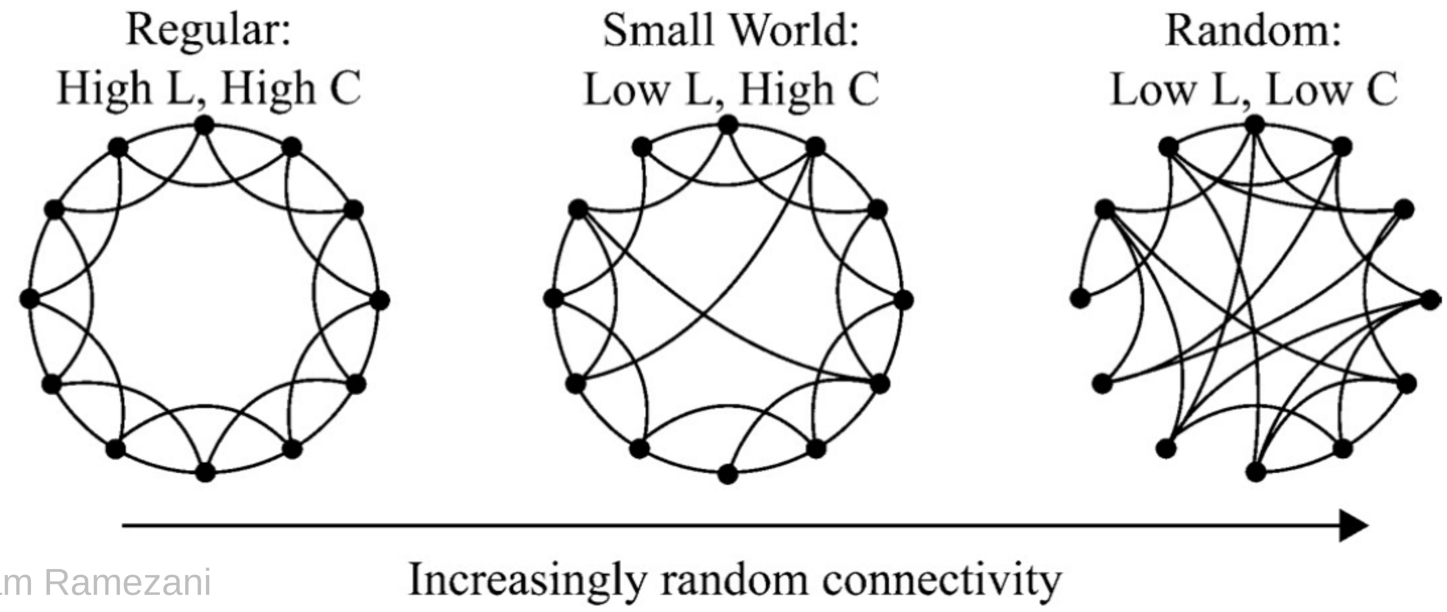
- **High clustering:** my friends' friends tend to be my friends
- **Short average paths**



Source: Watts, D.J., Strogatz, S.H.(1998) Collective dynamics of 'small-world' networks. Nature 393:440-442.

Watts-Strogatz Model

- The resulting graph is so that
  - for the value of P = 0 we will have the original ring graph
  - for the value of P = 1 produces a pure random graph
  - For some values of P between these two extremes the resulting network has small characteristics path length ,and at the same time, high clustering coefficient
  - the average degree will be <k> = 2m



Real-world networks

Network	size	Characteristic path length	Shortest path in fitted random graph	Clustering coefficient	Clustering in random graph
Film actors	225,226	3.65	2.99	0.79	0.00027
MEDLINE co-authorship	1,520,251	4.6	4.91	0.56	1.8×10^{-4}
E.Coli substrate graph	282	2.9	3.04	0.32	0.026
C.Elegans	282	2.65	2.25	0.28	0.05

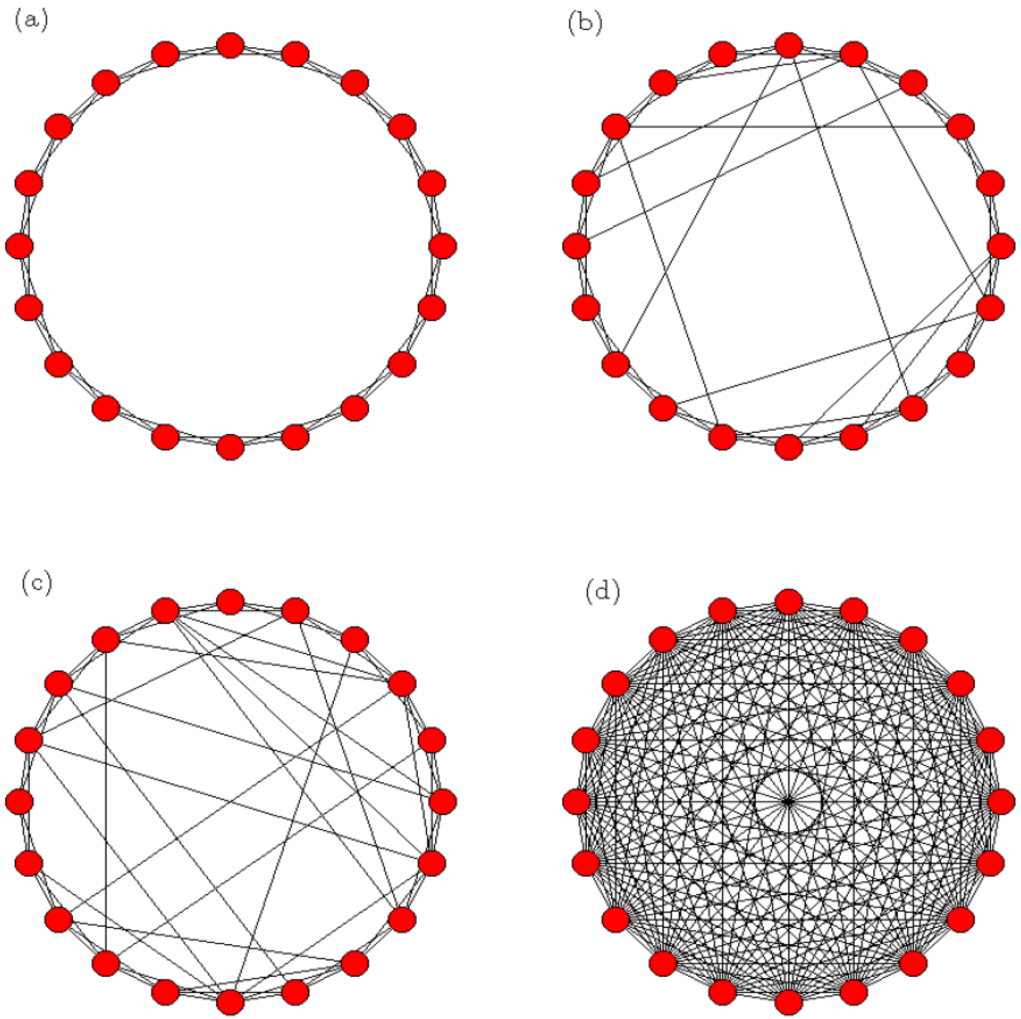
Newman-Watts model

- Starting with a k-ring graph
- N nodes
- Non-connected nodes get connected with probability P
- P = 1 results in complete graph
- for some small values of P
  - small-world property
  - high transitivity
- The networks are always connected

Newman-Watts model

20 nodes in a 2-regular ring with

a) $P = 0$
b) $P = 0.05$
c) $P = 0.15$
d) $P = 1$



04

Small-world Network

Newman-Watts model

- It was Longley believed that real-world networks have random structure
- Milgram did an experiment showing the small-world property
- Watts and Strogtaz showed that many real-world networks:
  - Have small characteristic path length compared to random networks
  - At the same time, have high clustering coefficient that is much larger than that of random networks
  - The are indeed small-worlds
- This discovery had huge impact on the various developments in Network fields
  - Search in complex networks
  - Communication in networks

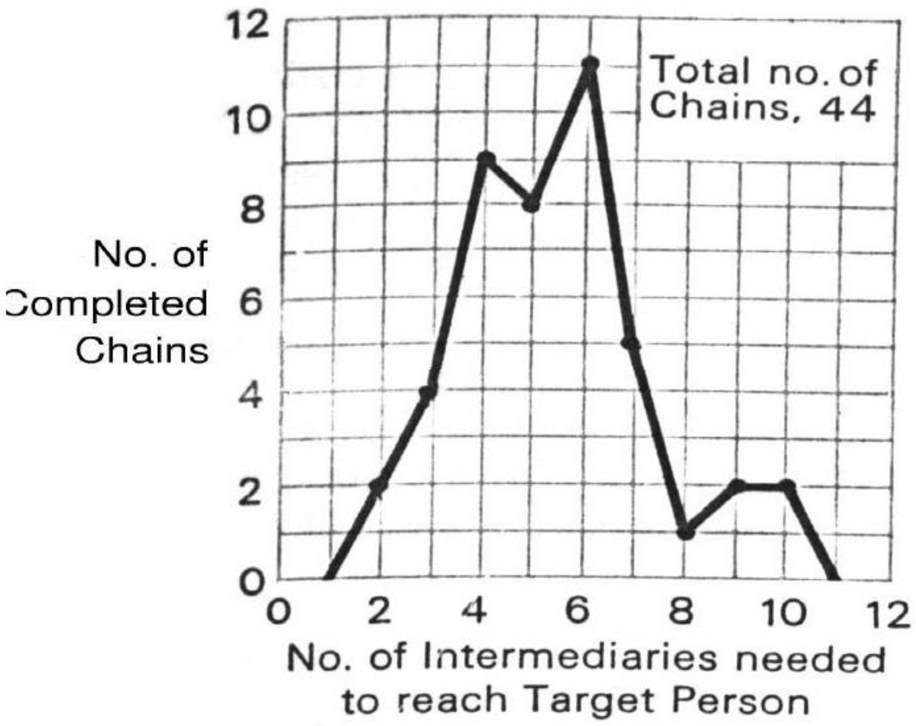
Milgram's experiment

- Instructions:
  - Given a target individual (stockbroker in Boston), pass the message to a person you correspond with who is "closest" to the target.
  - 160 letters: From Wichita (Kansas) and Omaha (Nebraska) to Sharon (Mass)
  - If you do not know the target person on a personal basis, do not try to contact him directly. Instead, mail this folder to a personal acquaintance who is more likely than you to know the target person.
- Outcome:
  - 20% of initiated chains reached
  - Target average chain length = 6.5
  - "Six degrees of separation"

Milgram, *Psych Today* **2**, 60 (1967)

Milgram's experiment

- "Six degrees of separation"
- The **Small World** concept in simple terms describes the fact despite their often **large size**, in most networks there is a **relatively short path between any two nodes**.



In the Nebraska Study the chains varied from two to 10 intermediate acquaintances with the median at five.

Milgram's experiment repeated

- Email experiment by Dodds, Muhamad, Watts, Science 301, (2003):
  - 18 targets
  - 13 different countries
  - More than 60,000 participants
  - 24,163 message chains
  - 384 reached their targets
  - Average path length 4.0

Source: NASA, U.S. Government; http://visibleearth.nasa.gov/view_rec.php?id=2429

Applicable to other networks?

Same pattern:

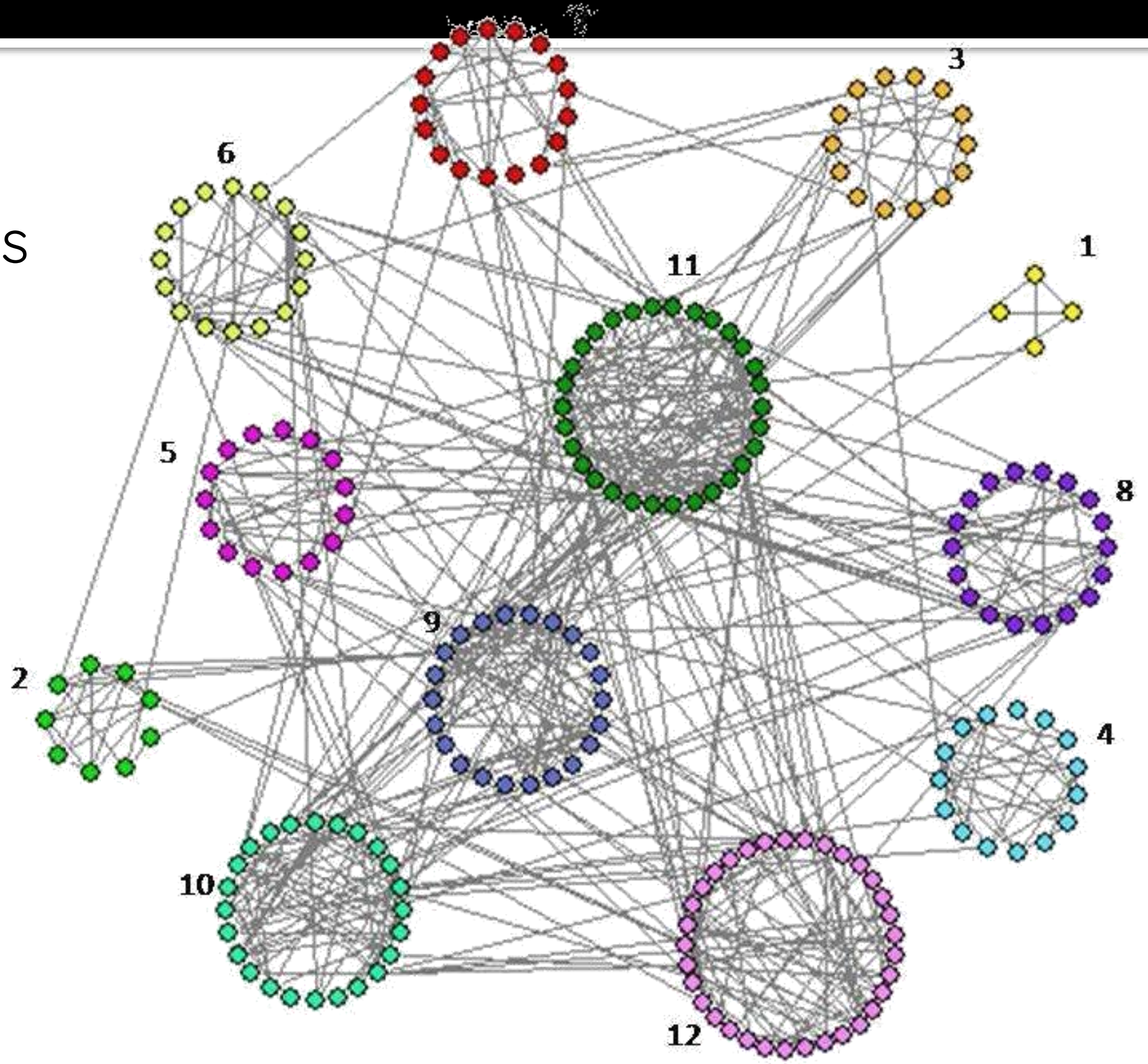
high clustering $$C_{\text{network}} >> C_{\text{randomgraph}}$$

low average shortest path $$l_{\text{network}} \approx \ln(N)$$

- of course in many social networks
- neural network of C. elegans,
- Human brain
- semantic networks of languages,
- actor collaboration graph
- food webs
- Power grids
- …

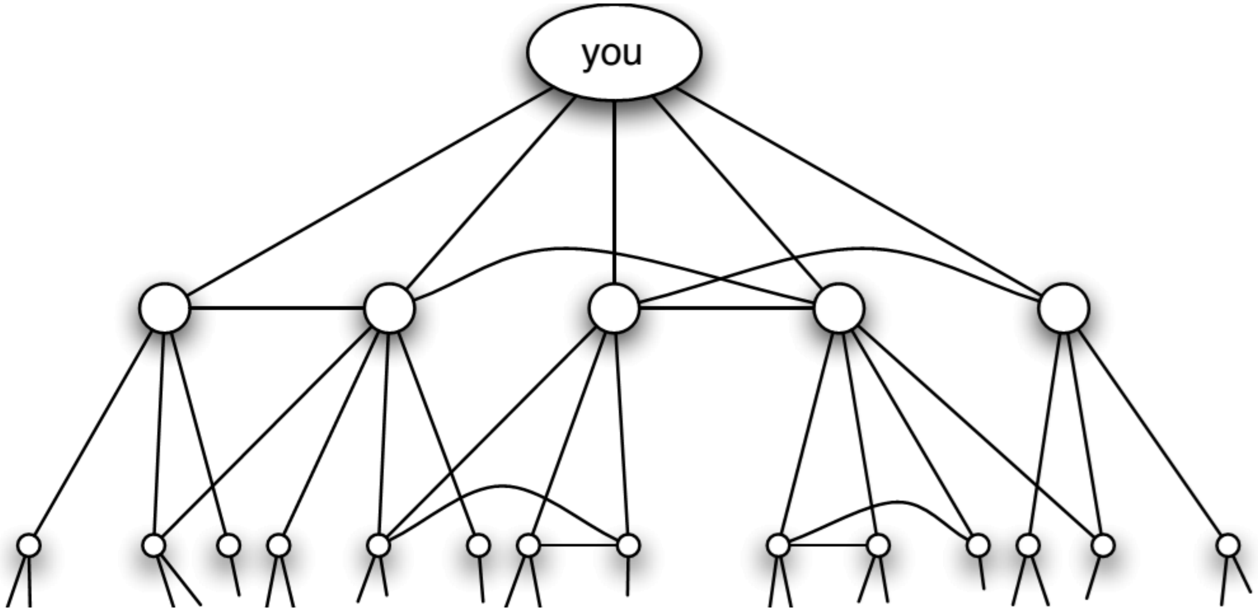
Small Worlds

- Six degrees of separation: although the number of edges is low, nodes are reachable from each other with small number of edges
- Small diameter or Small average path length
  - Weak ties to close dense communities
- Highly Clustered
  - High density of triangles
  - Homophily & prone to triadic closure



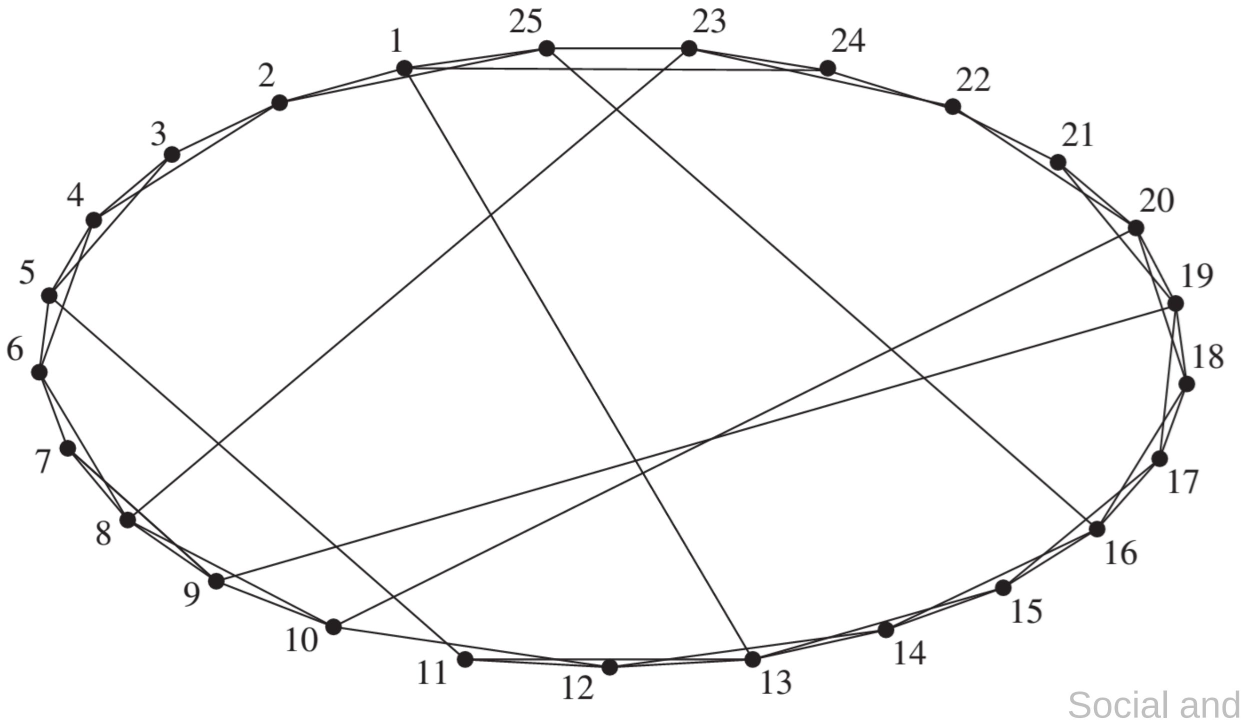
Structure + Randomness

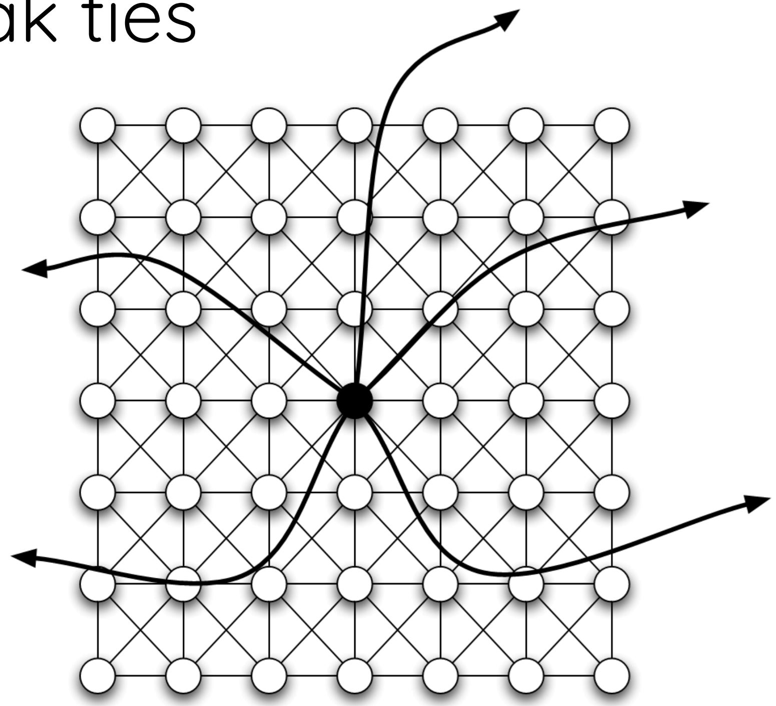
- Structure makes shortest paths
- Random links make triads
- It is naturally incorrect!



Structure + Randomness

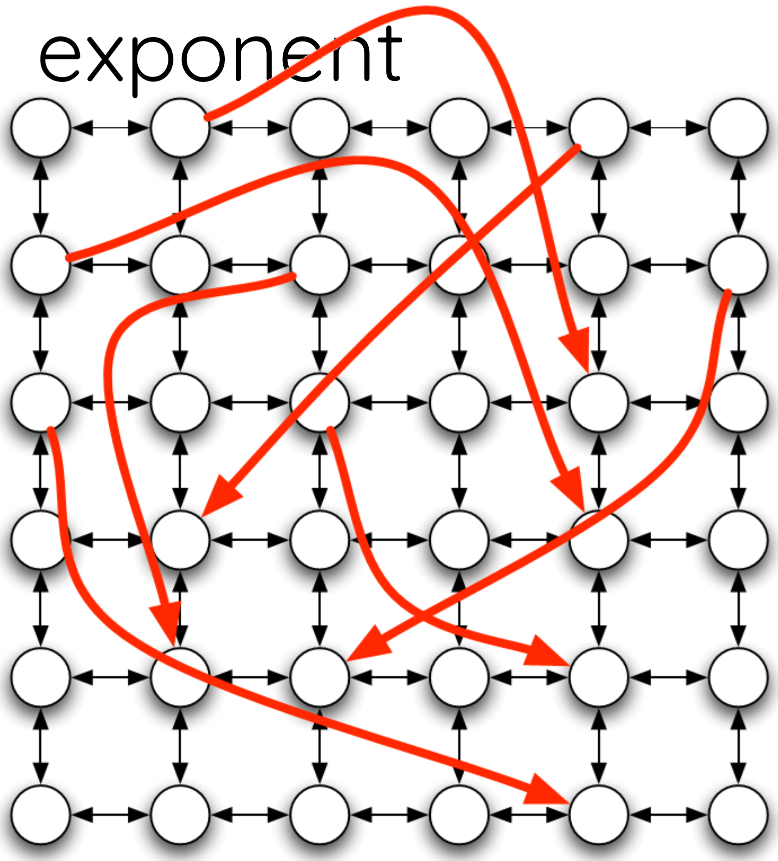
- Watts & Strogatz model
  - Structure makes triads
  - Random links make short distances: Weak ties

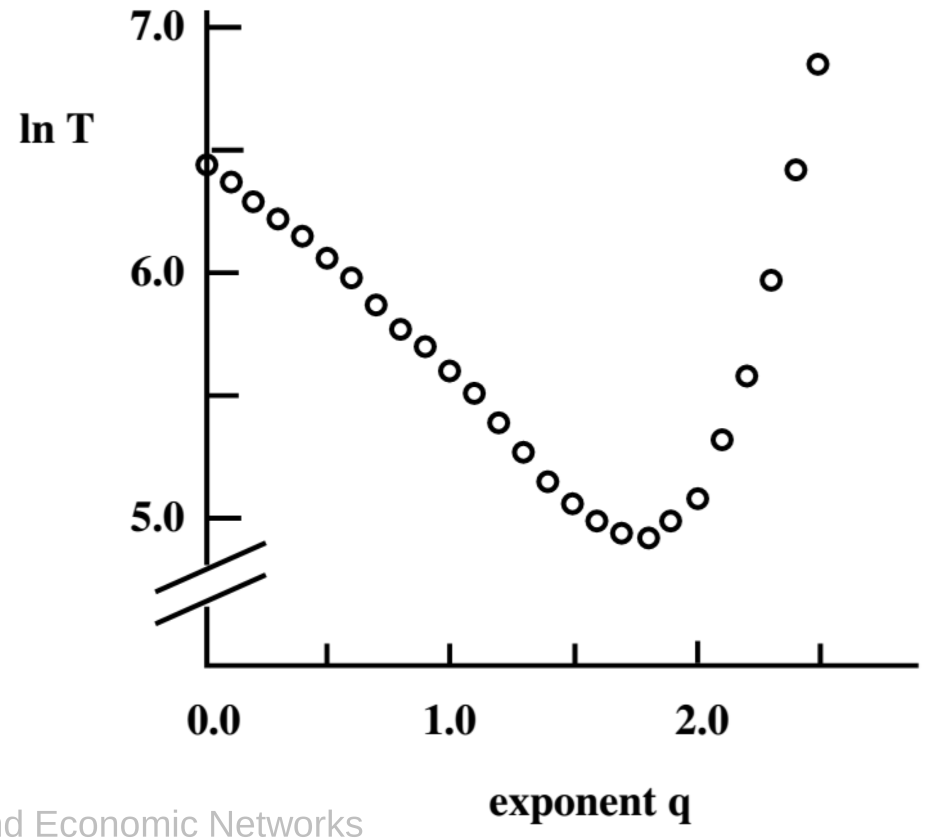




Watts-Strogatz Models for Decentralized Search

- Consider a grid with additional random links each with probability $d(v,w)^{-q}$ in which q is the clustering exponent

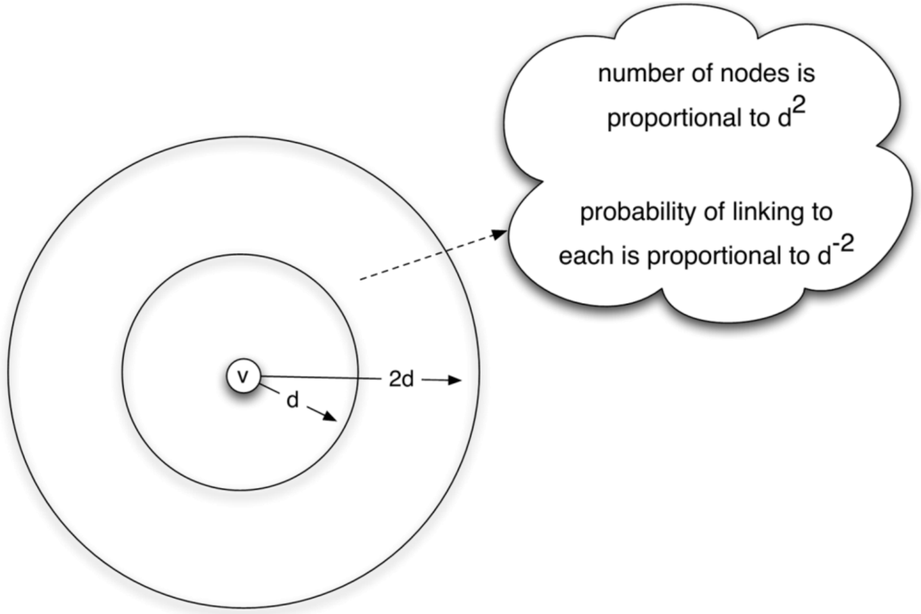




Watts-Strogatz Models for Decentralized Search

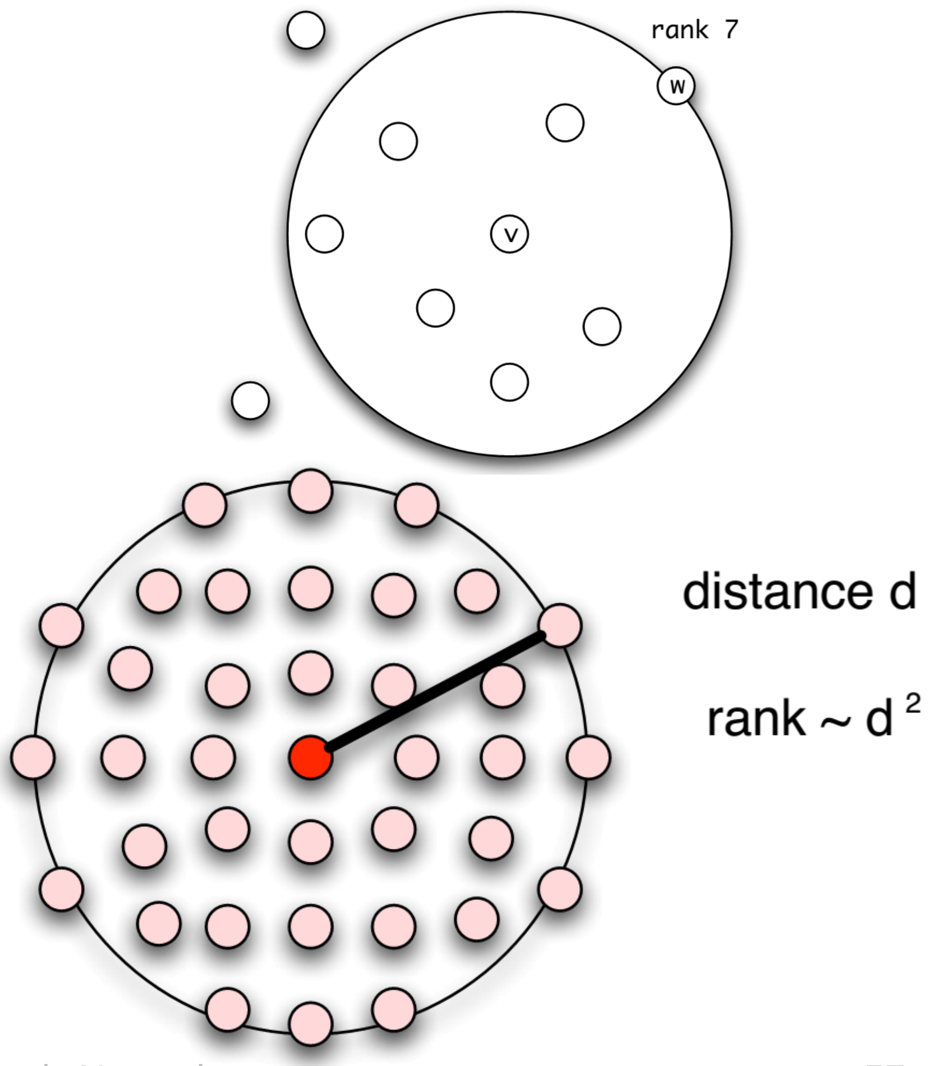
- Let's set the clustering coefficient q = 2
- Terms d^2 and d^{-2} cancel each other and thus the probability that a random edge links into *some node* in this ring is approximately independent of the value of *d*
- long-range weak ties are being formed in a way that's spread roughly uniformly over all different scales of resolution

$$\alpha\pi R^2 \times \frac{1}{R^2}$$



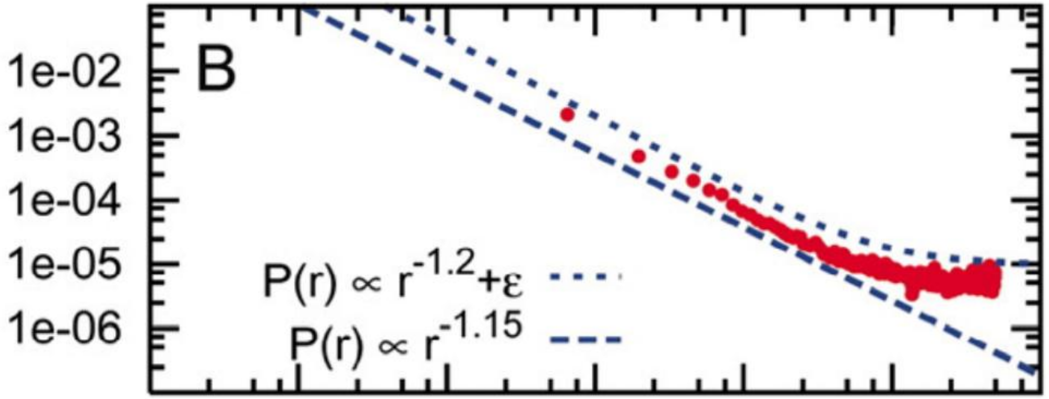
Watts-Strogatz Models for Decentralized Search

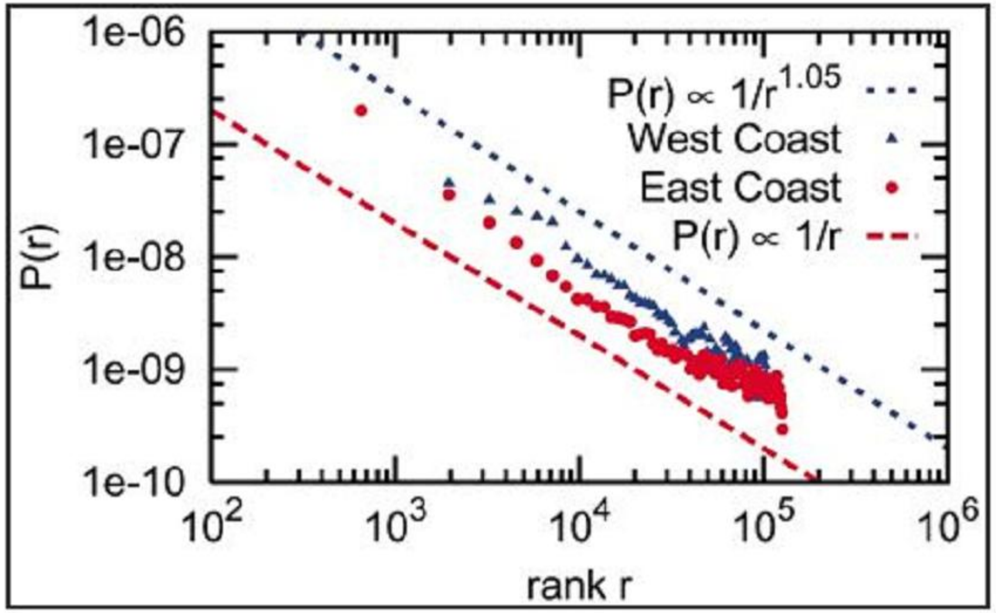
- Rank-based friendship:
  - Create (weak) random links with probability $rank(w)^{-p}$
  - What should p be to have a uniform spread of random links? $rank$ approximately is d^2, thus p should be approximately 1



Watts-Strogatz Models for Decentralized Search

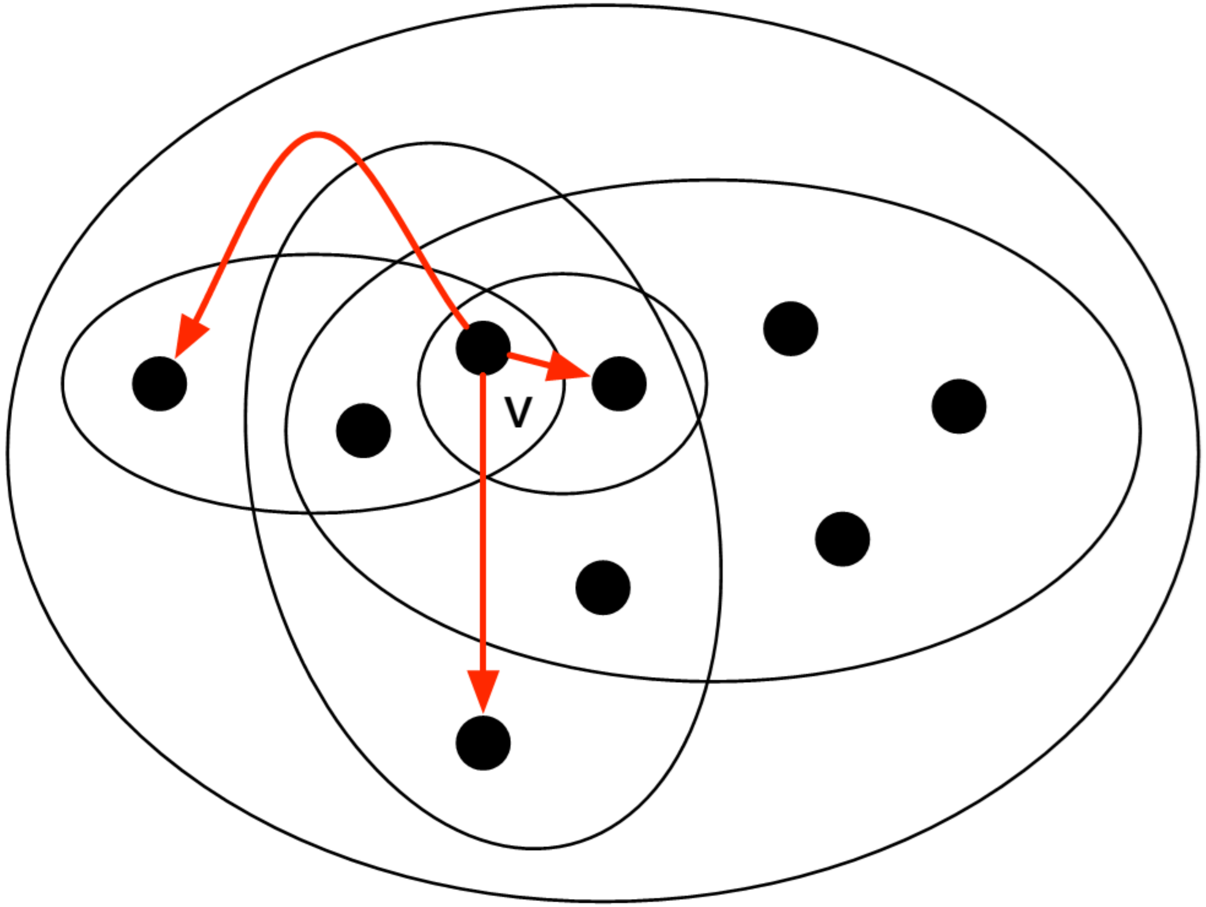
- Some Experiments





Watts-Strogatz Models for Decentralized Search

- Foci-based friendship:
  - Define the size of the smallest focal point that include both of v and w as their distance
  - We again draw random links with probability $dis(v,w)^p$
  - If focal points are defined as the nearest nodes, we may again have p = 1



Watts-Strogatz Models for Decentralized Search

- Mathematical study of myopic decentralized search in a simple Watts-Strogatz model:
  - A fixed structure: a ring or a grid with empty links
  - Some additional random links with probability proportional to $d(v,w)^{-1}$ with order of outdegree is 1
  - What is the constant multiplier for link probabilities:

$$Z \leq 2\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n/2}\right)$$

$$Z \leq 2+2\log_2(n/2) = 2+2(\log_2 n) - 2(\log_2 2) = 2\log_2 n$$

$$\frac{1}{Z}d(v,w)^{-1} \geq \frac{1}{2\log n}d(v,w)^{-1}$$

05

Markov Graphs & P* Networks

Markov Graphs & P* Networks

- Think about building a random graph in which the formation of the link ij is correlated with formation of the links jk and ik?
- Frank & Strauss method using Clifford & Hammersley theorem:
  - Build a graph D whose nodes is the potential links in G
  - If ij and jk are linked in D, it means that there exist some sort of dependency between them
  - C(D) is the set of D's cliques
  - $I_A(G) = 1$ for $A \in C(D)$, $A \subseteq G$ (consider G as a set of edges) and else $I_A(G) = 0$
  - The probability of a given network G depends only on which cliques of D it contains:

$$\log(\Pr[G]) = \sum_{A \in C(D)} \alpha_A I_A(G) - c$$

Markov Graphs & P* Networks

- An example: a symmetric case
  - Build a random graph with controllability on the number of its edges ($n_1(G)$) and its triads ($n_3(G)$)
  - C(D) consists of $n_3(G)$ triads and $n_1(G)$ edges. So, if we weight them equally, we have:
$$\log(\Pr(G)) = \alpha_1 n_1(G) + \alpha_3 n_3(G) - c$$
  - We can calibrate with different parameters to have different random networks with different number of triangles and edges.
    - $\alpha_3 = 0$ is the Poisson networks case

06

Configuration Model

The Configuration Model

- A sequence of degrees is given $(d_1, d_2, d_3, \dots, d_n)$ and we want to build a random graph having these degrees
- We generate the following sequence of numbers

$$\underbrace{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}_{d_1 \text{ entries}} \quad \underbrace{2, 2, 2, 2, 2, 2}_{d_2 \text{ entries}} \quad \dots$$

$$\underbrace{n, n, n, n, n, n, n, n, n, n, n, n}_{d_n \text{ entries}} .$$

- Randomly pick two number of elements and connect corresponding nodes
- The result is a multigraph

An Expected Degree Model

- Form a link between node *i* and node *j* with probability

$$p(e_{ij}) = \frac{d_i d_j}{\sum_k d_k} < 1$$

- Self links are allowed
- The expected degree of node *i* will be d_i
- Maximum of $d_i^2 < \sum_k d_k$

Configuration Model vs Expected Degree Model

- Consider the degree sequence $< k, k, \ldots, k >$
- In configuration model:
  - The probabilities of self links and multi links is negligible
  - The probability of a node to have degree k will converge to 1
- In expected degree model:
  - The probability of a node to have degree k will converge to

$$\frac{e^{-k}(k)^k}{k!}$$

whose maximum value is 1/2.

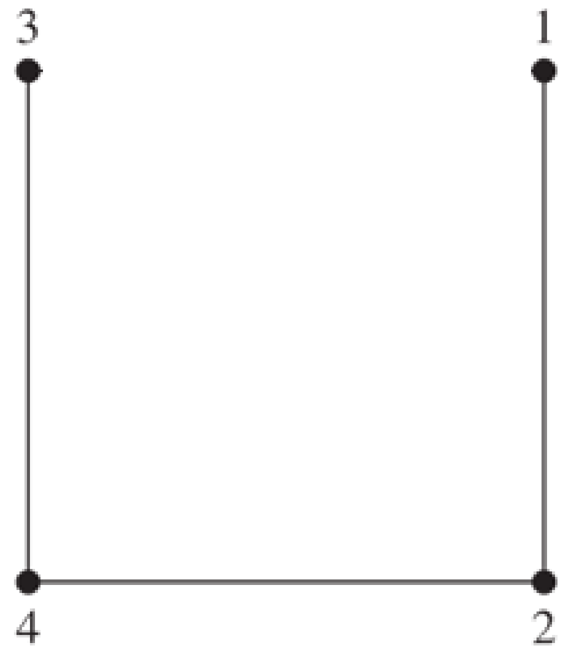
  - Why?

Distribution of the Degree of Neighboring Nodes

- Consider a given graph with degree distribution *P(d)*
- A related calculation $\tilde{P}(d)$: the probability that a randomly chosen edge has a (randomly chosen) neighbor with degree *d*
- $P(d) = \tilde{P}(d)$?
  - $P(1) = P(2) = \frac{1}{2}$
  - $\tilde{P}(1) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$
  - $\tilde{P}(2) = \frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{1} = \frac{2}{3}$
- We can formulate $\tilde{P}(d)$

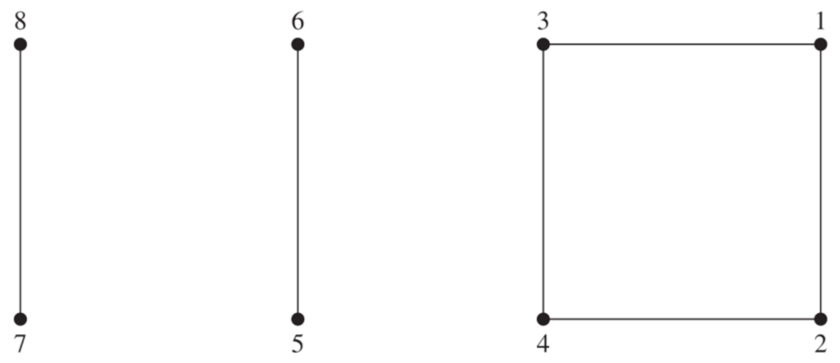
$$\tilde{P}(d) = \frac{P(d)d}{< d >}$$

See the blackboard



Distribution of the Degree of Neighboring Nodes

- Consider the degree sequence <1,1,2,2,1,1,2,2,...>. Compare two cases
  - In random models such as the configuration model: The distribution of the neighboring nodes have the same distribution as $\tilde{P}(d)$ for all nodes.
  - In networks with correlation properties: The graph is highly segregated by degrees



07

Preferential Attachment

Distribution of the Degree of Neighboring Nodes

- Consider the degree sequence <1,1,2,2,1,1,2,2,...>. Compare two cases
  - In random models such as the configuration model: The distribution of the neighboring nodes have the same distribution as $\tilde{P}(d)$ for all nodes.
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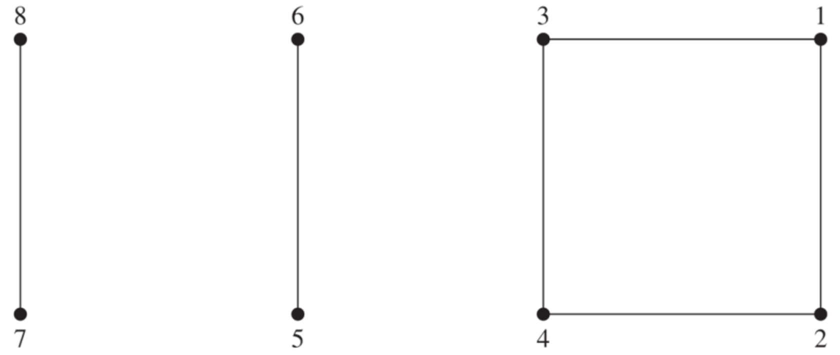
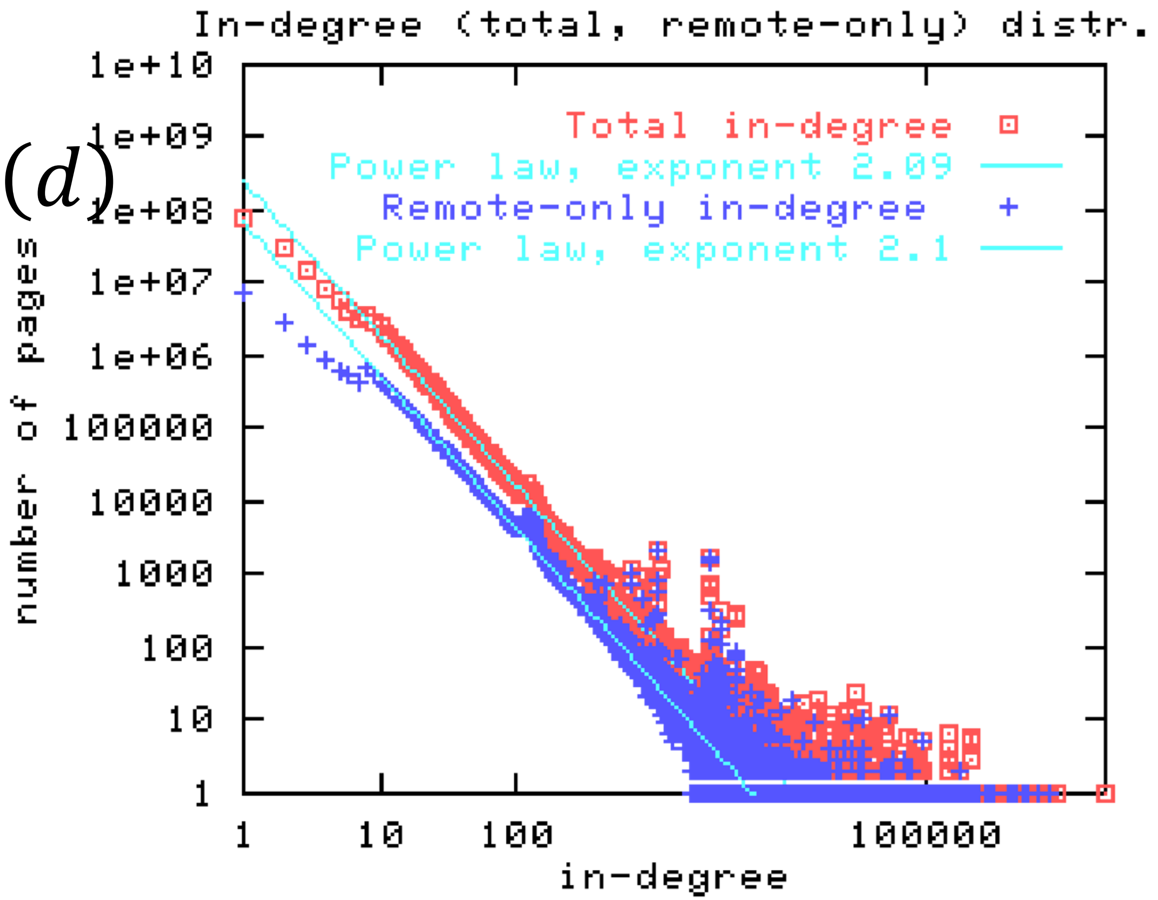


Power Law Degree Distribution

- $P(d) = cd^{-\gamma}$
- $\log(P(d)) = \log(c) - \gamma \log(d)$
- Features:
  - Scale-free
  - Fat tail



Richer-Get-Richer & Preferential Attachment

- In many scenarios, richers have more opportunity to get richers
  - More money for investment
  - Lower risks
  - More reputation to be involved in activities
  - ….
- Preferential Attachment: richer-get-richer effect in network creation
  - The probability that page *L* experiences an increase in popularity is directly proportional to *L*'s current popularity.
  - In the sense that links are formed "preferentially" to pages that already have high popularity

Preferential Attachment Models

- Devise models to simulate preferential attachment processes
- A basic growing model:
  - Nodes are born over time and indexed by their date of birth $i \in \{0, 1, 2 \ldots, t, \ldots\}$
  - Upon birth each new node forms m links with pre-existing nodes
  - It attaches to nodes with probabilities proportional to their degrees.
  - the probability that an existing node i receives a new link:

$$m \frac{d_i(t)}{\sum_{j=1}^{t} d_j(t)}$$

- The interesting fact is that these models leads to networks with power-law degree distribution

Growing Models

- A network model dealing with adding newborn nodes instead of statically having the whole network
- Consider a variation of the Poisson random setting
  - Start with a complete network of m+1 nodes
  - Each newborn node choose m nodes from the existing ones and links to them
- A natural study of degree distribution:
  - The expected degree of a node born at time i, at time t:

$$m + \frac{m}{i+1} + \frac{m}{i+2} + \cdots + \frac{m}{t} = m\left(1 + \frac{1}{i+1} + \cdots + \frac{1}{t}\right) \approx m\left(1 + \log\left(\frac{t}{i}\right)\right)$$

  - Degree distribution:

$$m\left(1 + \log\left(\frac{t}{i}\right)\right) < d \Rightarrow i > te^{1-\frac{d}{m}}$$

Growing Models

- A natural study of degree distribution:
  - The nodes with expected degree less than d are those born at time $te^{1-\frac{d}{m}}$
  - This is a fraction of $1 - e^{1-\frac{d}{m}}$ of total t nodes
  - Thus

$$F_t(d) = 1 - \mathrm{e}^{-\frac{d-m}{m}}$$

- Another way: Mean Field Approximation

Mean Field Approximation

- Using expected increase in the number of sth as its rate
- Visiting the last example with MFA:

$$\frac{dd_i(t)}{dt} = \frac{m}{t} \Rightarrow d_i(t) = m + m\log\left(\frac{t}{i}\right)$$

$$d = m + m\log\left(\frac{t}{i(d)}\right)$$

$$\frac{i(d)}{t} = e^{-\frac{d-m}{m}}$$

  - With the same argumentation we have:

$$F_t(d) = 1 - e^{-\frac{d-m}{m}}$$

Basic Preferential Attachment Model

- The probability that an existing node *i* receives a new link:

$$m\frac{d_i(t)}{\sum_{j=1}^{t} d_j(t)} = m\frac{d_i(t)}{2mt} = \frac{d_i(t)}{2t}$$

- Using MFA:

$$\frac{dd_i(t)}{dt} = \frac{d_i(t)}{2t}$$

- With initial condition $d_i(i) = m$ we have:

$$d_i(t) = m\left(\frac{t}{i}\right)^{\frac{1}{2}}$$

Basic Preferential Attachment Model

- We have:

$$\frac{i_t(d)}{t} = \left(\frac{m}{d}\right)^2$$

- Thus

$$F_t(d) = 1 - m^2 d^{-2} \Rightarrow f_t(d) = 2m^2 d^{-3}$$

- If the rate changes to $\frac{d_i(t)}{\gamma t}$ we have:

$$f_t(d) = \gamma m^\gamma d^{-\gamma-1}$$

Which is a power law distribution

Hybrid Preferential Attachment Models

- Mixing Random & Preferential Attachment:

$$\frac{dd_i(t)}{dt} = \frac{\alpha m}{t} + \frac{(1-\alpha)md_i(t)}{2mt} = \frac{\alpha m}{t} + \frac{(1-\alpha)d_i(t)}{2t}$$

- By solving the above differential equation we have:

$$d_i(t) = \phi_t(i) = \left(d_0 + \frac{2\alpha m}{1-\alpha}\right)\left(\frac{t}{i}\right)^{(1-\alpha)/2} - \frac{2\alpha m}{1-\alpha}$$

Hybrid Preferential Attachment Models

- By solving the above differential equation we have:

$$d_i(t) = \phi_t(i) = \left(d_0 + \frac{2\alpha m}{1-\alpha}\right)\left(\frac{t}{i}\right)^{(1-\alpha)/2} - \frac{2\alpha m}{1-\alpha}$$

- To have the degree distribution:
  - If $d_i(t) = \phi_t(i)$ (the degree of the node with i'th birth)

$$F_t(d) = 1 - \frac{\phi_t^{-1}(d)}{t}$$

$$\phi_t^{-1}(d) = t\left(\frac{d_0 + \frac{2\alpha m}{1-\alpha}}{d + \frac{2\alpha m}{1-\alpha}}\right)^{2/(1-\alpha)} \qquad F_t(d) = 1 - \left(\frac{m + \frac{2\alpha m}{1-\alpha}}{d + \frac{2\alpha m}{1-\alpha}}\right)^{2/(1-\alpha)}$$

Graph Properties

- A property P holds almost surely (or for almost every graph), if

$$\lim_{n\to\infty} P[G \text{ has } P] = 1$$

- Evolution of the graph: which properties hold as the probability p increases?
  - different from the evolving graphs that we will see in the future lectures
- Threshold phenomena: Many properties appear suddenly. That is, there exist a probability p_c such that for $p<p_c$ the property does not hold and for $p>p_c$ the property holds.

Any Question?